On optimal favoritism in all-pay contests

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Abstract: I analyze the optimal favoritism in a complete-information all-pay contest with two players, whose costs of effort are weakly convex. The contest designer could favor or harm some contestants using one of two instruments: head starts and handicaps. I find that any given player's effort distribution is ranked in the sense of first-order stochastic dominance according to how (ex post) symmetric the players are in terms of competitiveness. Consequently, as long as the designer values effort from both contestants, "leveling the playing field" is optimal regardless of which instrument is used.

Keywords: All-Pay Contests, Stochastic Dominance, Favoritism, Head Start, Handicap.

1 Introduction

Contests are widely used to allocate scarce resources among competing individuals. Examples include lobbying, college admissions, and competitions for job promotion opportunities (see

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Konrad [2009], Dechenaux et al. [2015], and Vojnović [2016]). In many situations, contestants are ex ante asymmetric in their abilities and positions. For instance, when considering a job promotion competition, the manager may notice differences in productivity levels and progress among employees. Therefore, she may want to tailor the competition to encourage more effort from employees.

Problems like this are commonplace: the contest organizer often has discretionary power in designing contest rules and takes advantage of them to induce more competition. Two general approaches are considered in the literature. One approach is to set individual-specific prizes, as in Gürtler and Kräkel [2010] and Pérez-Castrillo and Wettstein [2016], where the contest reward depends on the identity of the winner. Another approach is to set individual-specific contest success functions; in this case, when facing the same bidding profile, the two players have different probabilities of winning (see Drugov and Ryvkin [2017] and Fu and Wu [2020] for a general analysis).

This paper adopts the second approach and investigates the design of individual-specific contest success functions. Two commonly used instruments are considered: head starts, which are added to players' efforts, and handicaps, which discount players' efforts. In recent years, there has been growing literature exploring similar questions in various contest formats (see Konrad [2002], Epstein et al. [2011], Li and Yu [2012], Kirkegaard [2012], Franke et al. [2013], Seel and Wasser [2014], Kawamura and de Barreda [2014], and Franke et al. [2018]). The conventional wisdom suggests that it is optimal to "level the playing field": the contest designer prefers an unbiased contest when contestants are symmetric, but a biased contest favoring the weaker contestant when they are asymmetric. Nevertheless, in most of the aforementioned literature (with the exception of Kawamura and de Barreda [2014] and Drugov and Ryvkin [2017]), while costs are assumed to be linear with respect to effort, this assumption is not necessarily satisfied in many practical settings. For example, in a job promotion competition, much more effort is usually required to improve the work quality from excellent to perfect than from mediocre to good for any given employee.

From a theoretical perspective, this linearity assumption may cause us to overlook information that we should not have. Previous studies find the curvature of cost functions to be a decisive factor that cannot be ignored in contest design problems (see Moldovanu and Sela [2001], Drugov and Ryvkin [2017], Olszewski and Siegel [2020] and Fang et al. [2020]): Moldovanu and Sela [2001] find that it is optimal to allocate the entire prize to a single winner when cost functions are linear or concave; however, several positive prizes may be optimal when cost functions are instead convex. Similarly, Olszewski and Siegel [2020] find that the optimal number of prizes depends on the curvature of the costs in performance-maximizing large contests.

In light of this deficiency, the current paper investigates the optimal design of biased contests when cost functions are weakly convex. A related work is Drugov and Ryvkin [2017], which introduces a general class of biased contest success functions and studies optimal bias when players are ex ante symmetric. Drugov and Ryvkin [2017] provide conditions under which zero bias is optimal and prove through examples that biased contests may be optimal when such conditions fail. The key assumption in their model is that contest success functions are smooth: this includes Tullock [1980] lottery contests and Lazear and Rosen [1981] type tournaments. Nevertheless, one large class of contests is excluded by this assumption — that is, all-pay auctions, or more generally, all-pay contests.

In this paper, I use the framework in Siegel [2014a] and focus on the design of the optimal biased all-pay contest with complete information. A contest designer could influence the outcome of the contest by giving head starts to or handicapping players, whose costs of effort are weakly convex. For simplicity, I omit the word "weakly" hereafter: all relations are in the weak sense, unless explicitly stated as "strictly." I find that, regardless of which instrument is used, any given player's effort distribution is ranked in the sense of first-order stochastic dominance according to how (ex post) symmetric the two players are in terms of competitiveness. Consequently, for any objective functions increasing in efforts, it is optimal for the organizer to "level the playing field." Comparing head starts and handicaps, I show that no instrument always dominates over the other, and I provide sufficient conditions for head starts to be more efficient, under the two most-studied objectives: total effort and maximum individual effort. Lastly, I study optimal combinations of the two instruments under the aforementioned two objectives and find that the designer benefits from using both instruments simultaneously; in fact, by doing so, she achieves her first best result when her objective is maximum individual effort.

This paper's contributions are threefold. First, by allowing for convex cost functions, my result generalizes the conventional wisdom that a contest designer benefits from "leveling the playing field" and that (under certain conditions) head starts are a more efficient tool than handicaps. In this regard, a closely-related work is Li and Yu [2012], who find that, in revenue-maximizing all-pay auctions, "leveling the playing field" is optimal and that handicaps are less efficient than head starts. The current paper provides boundaries for their results to hold in more general cases: it may be suboptimal to "level the playing field" when cost functions are not convex, and sometimes handicaps can, in fact, be more efficient than head starts when cost functions are strictly convex.

Second, one conceptual element that sets this paper apart from the rest of the literature is that the current paper examines the distribution of efforts rather than a summary statistics, such as total effort. Consequently, I am able to show a stronger result: any player's effort distribution is ranked in the sense of first-order stochastic dominance, according to how (ex post) symmetric the two players are in their competitiveness.

Finally, some results in this paper also contribute to the contest literature by deepening our understanding of optimal favoritism in contests with non-smooth contest success functions under complete information. For example, one implication of my result is that in a symmetric all-pay auction with complete information, the optimal head start is of size zero. This contrasts with the findings in Seel and Wasser [2014] that, with incomplete information, the optimal head start is always of a strictly positive size. This discrepancy emphasizes the role that information plays in the design of all-pay contests and is consistent with findings on the design of the optimal handicaps in lottery contests (see Fu [2006] and Kirkegaard [2012] for analyses in complete and incomplete information settings respectively). Also, my model yields contrasting results to contest models with smooth contest success functions. For instance, Drugov and Ryvkin [2017] show that a head start can improve aggregate effort supply in two-player Tullock contests with two symmetric players. As Section 3.1 shows in the current paper, though, this result fails to hold when contest success functions are non-smooth.

The remainder of the paper is organized as follows. Section 2 sets up the model. Sections 3.1-3.3 identify the optimal head start and handicap and make a comparison between them. Section 3.4 discusses what happens when cost functions are not convex. Section 4 investigates the optimal combination of both instruments in two special cases, and Section 5 concludes. Omitted proofs are in the Appendix.

2 Model

There are two risk-neutral players and one contest designer.[1] The players, indexed by $i = 1, 2$, compete for a single prize by exerting efforts $e_i \geq 0$. Each player is characterized by her valuation of the prize, $V_i > 0$, and a cost function (of effort) $c_i : \mathbb{R}_+ \to \mathbb{R}_+$.

The contest designer is characterized by a utility function $\Lambda(e_1, e_2)$, and she can influence

[1] Fu and Wu [2020] show that in generalized lottery contests, "leveling the playing field" is optimal when there are two contestants, but not so when the number of contestants exceeds two.

the outcome of the contest with two instruments: head starts $a_i \geq 0$ and handicaps $h_i > 0$. Specifically, when player i exerts effort e_i, her score is $s_i = a_i + h_i e_i$. Denote by $\tilde{c}_i(s_i)$ player i's cost of achieving score s_i in the presence of head start a_i and handicap h_i. We have:

$$\tilde{c}_i(s_i) = \begin{cases} 0, & \text{if } s_i \leq a_i, \\ c_i(\frac{s_i - a_i}{h_i}), & \text{otherwise.} \end{cases}$$

Given $\mathbf{s} = (s_1, s_2)$, player i's payoff is

$$u_i(\mathbf{s}) = P_i(\mathbf{s})V_i - \tilde{c}_i(s_i),$$

where $P_i : \mathbb{R}^2_+ \to [0, 1]$ is player i's probability of winning, which satisfies

$$P_i(\mathbf{s}) = \begin{cases} 0, & \text{if } s_i < s_{-i}, \\ \text{any value in } [0,1], & \text{if } s_i = s_{-i}, \\ 1, & \text{if } s_i > s_{-i}, \end{cases}$$

such that $\sum_{i=1}^{2} P_i(\mathbf{s}) = 1$.

I make the following assumptions.

Assumption 1. $c_i(0) = 0$. *c_i is strictly increasing, differentiable, and weakly convex.*

Assumption 2. $\Lambda(e_1, e_2)$ *is increasing in both arguments.*

I normalize $V_i = 1$ for both players. This is without loss of generality. Extended from Siegel [2009], the equilibrium characterization uses the following definitions.

Definitions:

1. Player i's *reach* r_i is the maximum effort she can choose without obtaining a negative payoff if she wins the prize with certainty: $r_i = c_i^{-1}(1)$.[2] Re-index players such that $r_1 \geq r_2$.

2. Player i's *modified reach* $\tilde{r}_i$ is the maximum score she can obtain without incurring a negative payoff if she wins the prize with certainty, in the presence of handicaps and head starts: $\tilde{r}_i = \tilde{c}_i^{-1}(1)$.

3. The player with a lower modified reach is the *marginal player*.

4. The *threshold* T of the contest is the modified reach of the marginal player: $T = \min\{\tilde{r}_1, \tilde{r}_2\}$.

5. Player i's *power* w_i is her payoff when her score is T and wins: $w_i = 1 - \tilde{c}_i(T)$.[3]

The main departure from Siegel [2009] is the introduction here of favoritism in the contests and, as a result, modified reaches. According to Siegel [2009], in any equilibrium, the expected payoff of every player is equal to the maximum of her power and zero. In my setting, r_i determines player i's ex ante strength: without head starts or handicaps, the player with a higher reach obtains a positive equilibrium payoff and the player with a lower reach obtains an equilibrium payoff of zero. By contrast, in the presence of head starts and handicaps, $\tilde{r}_i = a_i + h_i r_i$ determines player i's ex post strength: the player with a higher modified reach obtains a positive equilibrium payoff, and the player with a lower modified reach obtains an equilibrium payoff of zero.

The difference between the two players' reaches, $r_1 - r_2$, measures ex ante how symmetric the two players are, and its counterpart, $\tilde{r}_1 - \tilde{r}_2$, measures ex post how symmetric the two players are, after the choice of head starts or handicaps (by the contest designer). In line

[2]Existence of r_i is implied by Assumption 1.

[3]I assume that $T - a_i \geq 0$. If this assumption is violated, the contest is so biased that neither player has an incentive to compete in equilibrium, which is a trivial case.

with existing literature, I say that a designer *levels the playing field* if players are ex post symmetric, namely $\tilde{r}_1 - \tilde{r}_2 = 0$.

With two players, the model can be simplified because only the relative sizes of the two players' head starts and handicaps matter. It is without loss of generality to normalize Player 1's head start a_1 to zero and handicap h_1 to one and to focus on instruments placed on Player 2. Namely, players' efforts are mapped to scores as: $s_1 = e_1$ and $s_2 = a + he_2$, where $a \in \mathbb{R}$ and $h \in \mathbb{R}_+$. $a > 0$ means that Player 2 is given a head start, and $a < 0$ means that Player 1 is given a head start.[4] Likewise, $h < 1$ means that Player 2 is handicapped, and $h > 1$ means that Player 1 is handicapped.

Lastly, for tractability, most of the following analysis is focused on the optimal use of a single instrument with the exception of Section 4, which explores the optimal combination of two instruments when the contest designer's objective is maximum individual effort.

3 Optimal Favoritism with a Single Instrument

3.1 Optimal Head Start

When the contest designer only uses head starts as an instrument, namely $h = 1$, the size of head starts falls into one of the three cases:

Case 1: $a \leq 0$. A head start is given to the ex ante stronger player and makes her even stronger. By definition, $T = r_2$, $w_1 = 1 - c_1(T + a)$ and $w_2 = 0$.

Case 2: $r_1 - r_2 < a$. A large head start is given to the ex ante weaker player such that ex

[4] This negative a is different from a nagative head start $(-\beta)$ given to Player 2 in Drugov and Ryvkin [2017]. In my model, only the difference in the two player's head starts matters. Consequently, all results remain if a negative head start for each individual is allowed ($a_i \in \mathbb{R}$).

post she becomes the stronger player. By definition, $T = r_1$, $w_1 = 0$ and $w_2 = 1 - c_2(T - a)$.

Case 3: $0 < a \leq r_1 - r_2$. A small head start is given to the ex ante weaker player such that ex post she is still the weaker player. By definition, $T = r_2 + a$, $w_1 = 1 - c_1(T)$ and $w_2 = 0$.

Below I first solve for the unique equilibrium and then show the first-order stochastic dominance (FOSD hereafter) property of players' effort distributions. I will illustrate the methodology in detail with Case 1 in which a head start is given to Player 1, and present its result in Proposition 1. Proposition 2 is the parallel result when a head start is given to Player 2, which corresponds to Cases 2 and 3.

Following Siegel [2014a], the unique equilibrium is in mixed strategies. Moreover, in any equilibrium, the expected payoff of every player is equal to the maximum of her power and zero.[5] Let $G_i^{HS}(e;a)$ denote the CDF of player i's equilibrium strategy, which specifies player i's probability of choosing efforts less than or equal to e, when the size of the head start is a. For simplicity, I call it player i's *effort distribution induced by* a. Recall that $w_1 = 1 - c_1(T + a)$ and $w_2 = 0$, which are players' equilibrium payoffs. According to the algorithm proposed in Siegel [2014a], this payoff characterization pins down $G_i^{HS}(e;a)$ for $a \leq 0$:

$$G_1^{HS}(e;a) = \begin{cases} c_2(e-a), & \text{if } e \in [0, T+a), \\ 1, & \text{if } e \in [T+a, \infty), \end{cases} \tag{1}$$

$$G_2^{HS}(e;a) = \begin{cases} w_1, & \text{if } e \in [0, -a), \\ c_1(e+a) + w_1, & \text{if } e \in [-a, T), \\ 1, & \text{if } e \in [T, \infty). \end{cases} \tag{2}$$

[5]When $a = r_1 - r_2$, the Power Condition (Assumptions C3(i) and M3(i)) in Siegel [2014a] is not met. Siegel [2014b] has shown that in contests with two players, the payoff result holds even if the Power Condition fails. Notice also that the algorithm and its uniqness does not rely on the power condition directly.

Proposition 1. *When $a' < a \leq 0$, $G_i^{HS}(\cdot; a)$ FOSD $G_i^{HS}(\cdot; a')$, i.e. $G_i^{HS}(e; a) \leq G_i^{HS}(e; a')$ for $i = 1, 2$ and $e \geq 0$. In particular, for all $i = 1, 2$ and $a \leq 0$, $G_i^{HS}(\cdot; 0)$ FOSD $G_i^{HS}(\cdot; a)$.*

Proof.

$$\frac{\partial G_1^{HS}(e;a)}{\partial a} = \begin{cases} -c_2'(e-a) < 0 & \text{if } e \in [0, T+a), \\ 0 & \text{if } e \in [T+a, \infty), \end{cases}$$

$$\frac{\partial G_2^{HS}(e;a)}{\partial a} = \begin{cases} -c_1'(T+a) < 0 & \text{if } e \in [0, -a), \\ c_1'(e+a) - c_1'(T+a) \leq 0 & \text{if } e \in [-a, T), \\ 0 & \text{if } e \in [T, \infty). \end{cases}$$

This shows that $G_i^{HS}(\cdot; a)$ FOSD $G_i^{HS}(\cdot; a')$ in the interior of each interval.

FOSD of G_i^{HS} holds on the "boundaries" as well. Consider a change in the size of the head start from a to $a-\delta$ for small $\delta > 0$. When $e \in [T+a-\delta, T+a]$, $G_1^{HS}(e; a)$ and $G_1^{HS}(e; a-\delta)$ are on different segments in (1). Since $G_1^{HS}(e; a) = c_2(e-a) \leq 1 = G_1^{HS}(e; a-\delta)$, FOSD of $G_1^{HS}(\cdot)$ holds on the boundary.

Likewise, $G_2^{HS}(e; a)$ and $G_2^{HS}(e; a-\delta)$ are on different segments in (2) when $e \in [-a, -a+\delta]$. It is less straightforward to prove $G_2^{HS}(e; a) \leq G_2^{HS}(e; a-\delta)$. Recall that $G_2^{HS}(e; a) = c_1(e+a) + 1 - c_1(T+a)$ and $G_2^{HS}(e; a-\delta) = w_1 = 1 - c_1(T+a-\delta)$. Notice that $G_2^{HS}(e; a-\delta)$ is constant on $e \in [-a, -a+\delta]$ and that $G_2^{HS}(e; a)$ is increasing in e, it suffices to prove the inequality at $e = -a+\delta$. Let $\Gamma(\delta)$ denote the difference between $G_2^{HS}(e; a)$ and $G_2^{HS}(e; a-\delta)$ at $e = -a+\delta$. That is,

$$\begin{aligned} \Gamma(\delta) &= G_2^{HS}(-a+\delta; a) - G_2^{HS}(-a+\delta; a-\delta) \\ &= c_1(\delta) - c_1(T+a) + c_1(T+a-\delta). \end{aligned}$$

$\Gamma(\delta) \leq 0$ for all small δ because $\Gamma(0) = 0$ and $\Gamma'(\delta) = c_1'(\delta) - c_1'(T + a - \delta) \leq 0$. Therefore, FOSD of $G_2^{HS}(\cdot)$ holds on the boundary. This completes the proof. $\square$

In the proof, we see that as the size of head start given to Player 1 decreases, both players adopt more aggressive strategies in equilibrium in the sense of first-order stochastic dominance. There is, however, a subtle difference. Player 1 becomes more aggressive as long as c_2 is increasing. The main reason is that Player 2 obtains a constant equilibrium payoff of zero. A smaller head start given to Player 1 together with an increasing c_2 implies that Player 2 now competes with Player 1 at a lower cost. To keep Player 2's payoff constant, Player 1 becomes more aggressive.

The analogous property of Player 2's strategy, however, requires more assumptions on c_1. This is because Player 1's equilibrium payoff w_1 decreases as a result of a smaller head start. On one hand, this decrease implies that Player 2 now competes more aggressively. On the other hand, similar to the previous analysis, a smaller head start given to Player 1 together with an increasing c_1 implies that Player 1 will now compete with Player 2 at a higher cost. Thus, Player 2 becomes less aggressive. The sign of the overall effect, captured by $c_1'(e + a) - c_1'(T + a)$, is determined by the curvature of c_1.

Combining the analyses above, Proposition 1 concludes that the contest designer would be worse off if she gives a strictly positive head start to the ex ante stronger player. As the following proposition suggests, this idea of "leveling the playing field" also applies when the designer gives a head start to the ex ante weaker player.

Proposition 2. *When $a' > a \geq r_1 - r_2$, $G_i^{HS}(\cdot; a)$ FOSD $G_i^{HS}(\cdot; a')$. When $r_1 - r_2 \geq a' > a \geq 0$, $G_i^{HS}(\cdot; a')$ FOSD $G_i^{HS}(\cdot; a)$. Therefore, $G_i^{HS}(\cdot; r_1 - r_2)$ FOSD $G_i^{HS}(\cdot; a)$ for $i = 1, 2$ and $a \geq 0$.*

Lemma 1 combines Propositions 1 and 2 and shows that the equilibrium effort distributions induced by $r_1 - r_2$ dominate over those induced by any other head starts.

Lemma 1. *$G_i^{HS}(\cdot; r_1 - r_2)$ FOSD $G_i^{HS}(\cdot; a)$ for any $a \in \mathbb{R}_+$.*

Proof. When $a \leq 0$, according to Proposition 1, $G_i^{HS}(\cdot; a)$ is dominated by $G_i^{HS}(\cdot; 0)$, which by Proposition 2 is dominated by $G_i^{HS}(\cdot; r_1 - r_2)$. Proposition 2 shows that $G_i^{HS}(\cdot; r_1 - r_2)$ FOSD $G_i^{HS}(\cdot; a)$ when $a > 0$. This completes the proof. □

Because $G_1^{HS}(\cdot; a)$ and $G_2^{HS}(\cdot; a)$ are independent, FOSD properties in Lemma 1 imply that the optimal head start is $r_1 - r_2$. Theorem 1 is the main result in this section.

Theorem 1. *The optimal head start, denoted by a^*, is $r_1 - r_2$.*

Proof. To show that $a^* = r_1 - r_2$ is optimal, consider any head start a. By Lemma 1, $G_i^{HS}(e; a^*) \leq G_i^{HS}(e; a)$. Let $v_i(e) = \inf\{y : G_i^{HS}(y; a^*) \geq G_i^{HS}(e; a)\}$. Then,

$$\begin{aligned}
\iint \Lambda(e_1, e_2) \mathrm{d}G_1^{HS}(e_1; a^*) \mathrm{d}G_2^{HS}(e_2; a^*) &\geq \iint \Lambda(v_1(e_1), v_2(e_2)) \mathrm{d}G_1^{HS}(e_1; a) \mathrm{d}G_2^{HS}(e_2; a) \\
&\geq \iint \Lambda(e_1, e_2) \mathrm{d}G_1^{HS}(e_1; a) \mathrm{d}G_2^{HS}(e_2; a),
\end{aligned}$$

where the first inequality is by substitution and the definition of $v_i(e)$ and the second inequalities is by the fact that $\Lambda(e_1, e_2)$ is increasing and that $v_i(e_i) \geq e_i$, which is true because $G_i^{HS}(e; a^*) \leq G_i^{HS}(e; a)$. □

Theorem 1 shows that giving a head start of size $r_1 - r_2$ to Player 2, namely "leveling the playing field", maximizes the expectation of any increasing utility functions, in which $\Lambda(e_1, e_2) = e_1 + e_2$ and $\Lambda(e_1, e_2) = \max\{e_1, e_2\}$ are included as special cases. As an implication of Theorem 1, Corollaries 1 and 2 consider these two utility functions and characterize optimal head starts in an ex ante asymmetric contest and in an ex ante symmetric contest, respectively.

Corollary 1. *In an ex ante asymmetric contest, providing a head start of size $r_1 - r_2$ to Player 2 maximizes both expected total effort and the expected maximum individual effort.*

Corollary 2. *In an ex ante symmetric contest, zero head start maximizes both the expected total effort and the expected maximum individual effort.*

3.2 Optimal Handicap

When the contest designer only uses handicaps as an instrument, namely $a = 0$, the size of the handicap falls into one of the three cases:

Case 1: $h \leq 1$. The ex ante weaker player is handicapped and hence becomes even weaker. By definition, $T = r_2 h$, $w_1 = 1 - c_1(T)$ and $w_2 = 0$.

Case 2: $\frac{r_1}{r_2} < h$. The ex ante stronger player is handicapped such that she becomes the ex post weaker player. By definition, $T = \frac{r_1}{h}$, $w_1 = 0$ and $w_2 = 1 - c_2(T)$.

Case 3: $1 < h \leq \frac{r_1}{r_2}$. Although the ex ante stronger player is handicapped, she is still ex post stronger than her opponent. By definition, $T = r_2$, $w_1 = 1 - c_1(hT)$ and $w_2 = 0$.

Let $G_i^{HC}(e; h)$ denote the CDF of player i's equilibrium strategy when the size of the handicap is h. The analysis of the optimal handicap is similar to that of the optimal head start. Hence I list below the corresponding propositions and the main result, while leaving details in the Appendix.

Proposition 3. *When $h' > h \geq \frac{r_1}{r_2}$, $G_i^{HC}(\cdot; h)$ FOSD $G_i^{HC}(\cdot; h')$. When $\frac{r_1}{r_2} > h' > h \geq 1$, $G_i^{HC}(\cdot; h')$ FOSD $G_i^{HC}(\cdot; h)$. Therefore, $G_i^{HC}(\cdot; \frac{r_1}{r_2})$ FOSD $G_i^{HC}(\cdot; h)$ for $i = 1, 2$ and $h \geq 1$.*

Proposition 4. *When $h' < h \leq 1$, $G_i^{HC}(\cdot; h)$ FOSD $G_i(\cdot; h')$. In particular, for $i = 1, 2$ and $h \leq 1$, $G_i^{HC}(\cdot; 1)$ FOSD $G_i^{HC}(\cdot; h)$.*

Lemma 2. *$G_i^{HC}(\cdot; \frac{r_1}{r_2})$ FOSD $G_i^{HC}(\cdot; h)$ for any $h > 0$.*

Theorem 2. *The optimal handicap, denoted by h^*, is $\frac{r_1}{r_2}$.*

Corollary 3. *In an ex ante asymmetric contest, handicapping Player 1 by the size of $\frac{r_2}{r_1}$ maximizes both expected total effort and the expected maximum individual effort.*

Corollary 4. *In an ex ante symmetric contest, zero handicapping maximizes both the expected total effort and the expected maximum individual effort.*

Remarks: all results in this section holds when $ec_i'(e)$ is increasing in e, which is implied by convexity of c_i. More discussion about this can be found in Section 3.4.

3.3 Comparison between Instruments

If the contest organizer could choose between the two instruments, which one would she choose? Proposition 5 shows that in general there is no definite answer: Player 1 exerts more efforts when head starts are used instead of handicaps; however, Player 2 exerts less efforts in this case.

Proposition 5. *$G_1^{HS}(\cdot; a^*)$ FOSD $G_1^{HC}(\cdot; h^*)$, but $G_2^{HC}(\cdot; h^*)$ FOSD $G_2^{HS}(\cdot; a^*)$.*

Proposition 5 implies that a contest designer who values both players' efforts does not always prefer one instrument to the other. This is illustrated with Examples 1 and 2 below with two widely used objective functions: $\Lambda(e_1, e_2) = e_1 + e_2$ and $\Lambda(e_1, e_2) = \max\{e_1, e_2\}$.

Example 1. Suppose that $\Lambda(e_1, e_2) = e_1 + e_2$.

When $c_1(e) = \frac{1}{2}e^2$ and $c_2(e) = e^2$, reaches are $r_1 = \sqrt{2}$ and $r_2 = 1$. Optimal instruments are $a^* = \sqrt{2} - 1$ and $h^* = \sqrt{2}$. The expected total effort induced by the optimal head start and the optimal handicap are $\sqrt{2}+\frac{1}{\sqrt{2}}-\frac{1}{2}\approx 1.6213$ and $\frac{2}{3}+\frac{2}{3\sqrt{2}}\approx 1.6095$, respectively. The contest organizer therefore prefers "leveling the playing field" with head starts.

When $c_1(e)=\frac{1}{2}e^2$ and $c_2(e)=e^3$, reaches and optimal instruments stay unchanged. The expected total effort induced by the optimal head start and the optimal handicap are now $\sqrt{2}+\frac{1}{\sqrt{2}}-\frac{5}{12}\approx 1.7047$ and $\frac{2}{3}+\frac{3}{2\sqrt{2}}\approx 1.7273$, respectively. The contest organizer therefore prefers "leveling the playing field" with handicaps. $\square$

Example 2. Suppose that $\Lambda(e_1,e_2)=\max\{e_1,e_2\}$.

When $c_1(e)=e$ and $c_2(e)=\frac{9}{8}e$, reaches are $r_1=1$ and $r_2=\frac{8}{9}$. Optimal instruments are $a^*=\frac{1}{9}$ and $h^*=\frac{9}{8}$. The expected maximum individual effort induced by the optimal head start and the optimal handicap are $\frac{2503}{3888}\approx 0.6438$ and $\frac{307}{486}\approx 0.6317$, respectively. The contest organizer therefore prefers "leveling the playing field" with head starts.

When $c_1(e)=e^3$ and $c_2(e)=\frac{9}{8}e$, reaches and optimal instruments stay unchanged. The expected maximum individual effort induced by the optimal head start and the optimal handicap are now $\frac{24056}{32805}\approx 0.7333$ and $\frac{199}{270}\approx 0.7370$, respectively. The contest organizer therefore prefers "leveling the playing field" with handicaps. $\square$

Despite the lack of a general ranking result, Propositions 6 and 7 below provide sufficient conditions under which head starts are preferred.

Proposition 6. *If $c_1(e)\le c_2(\frac{e}{h^*})$ for all $e\in[0,r_1]$, then the contest with the optimal head start induces higher expected total effort than that with the optimal handicap.*

Proposition 7. *If there is a function $\phi(\cdot):[0,r_1r_2]\to\mathbb{R}_+$ and $m\ge 0$ such that for all $e_1\in[0,r_1]$ and $e_2\in[0,r_2]$, $c_1(e_1)c_2(e_2)=\phi(e_1e_2)e_2^m$, then the contest with the optimal head start induces higher expected maximum individual effort than that with the optimal handicap.*

Proposition 6 suggests that when c_1 is lower than c_2 even if the handicap is counted in, the optimal head start induce higher expected total effort than the handicap does. With a much lower cost, Player 1's effort is of higher magnitude compared with Player 2's. Recall that when head starts are used instead of handicaps, Player 1's expected effort gets higher, but Player 2's expected effort gets lower. The increase in Player 1's expected effort, therefore, dominates the decrease in Player 2's, leading to an increase in total effort. The condition in Proposition 6 is quite general. For example, it is satisfied when c_1 is a scale down of c_2, namely $c_1(e) = \tau_0 c_2(e)$ for some $\tau_0 < 1$. This includes linear cost functions like the ones in Li and Yu [2012], as special cases.

Proposition 7 suggests that when c_2 is of higher degrees than c_1, the optimal head start is more efficient when the organizer's objective is maximum individual effort. The main reason is that the distribution of an order statistic is the multiplication of two players' equilibrium strategies. A sufficient condition for head starts to be more efficient is $c_1(e+a^*)c_2(e-a^*) \leq c_1(eh_1^*)c_2(\frac{e}{h_1^*})$, which holds when the condition in Proposition 7 is met. This condition is also quite general. For example, it is satisfied by monomial cost functions with $\mathrm{Deg}(c_1) \leq \mathrm{Deg}(c_2)$. This includes linear cost functions, which are widely studied in the literature, as special cases.

3.4 Non-Convex Cost Functions

What would happen if cost functions are not convex? Example 3 shows that, in this case, "leveling the playing field" with head starts may not be optimal.

Example 3. In a contest with two ex ante symmetric players with costs $c_i(e) = \sqrt[3]{e}$. Reaches for players are both 1. Consider giving Player 2 a head start of size $a \in [0, 1)$. In equilibrium, expected efforts of players 1 and 2 are $W_{01}(a) = \frac{\sqrt[3]{1-a}}{4}(3a+1)$ and $W_{02}(a) = \frac{1}{4}\left(3a^{4/3} - 4a + 1\right)$ and respectively. Figure 1 depicts both individual efforts and total effort. Expected total effort $W_0(a)$ attains its maximum at $a^* \approx 0.5688$.

Let $W_1(a)$ denote the expected maximum individual effort. $W_1(a) = \frac{1}{4}(-3(\sqrt[3]{1-a}\,-1)a^{4/3} + (7\sqrt[3]{1-a} - 4)a + 1)$ is maximized at $a^* \approx 0.6546$, as in Figure 2.

Both maximizers for expected total effort and the expected maximum individual effort are strictly positive when two players are ex ante symmetric. Hence "leveling the playing field" is not optimal in this setting. □

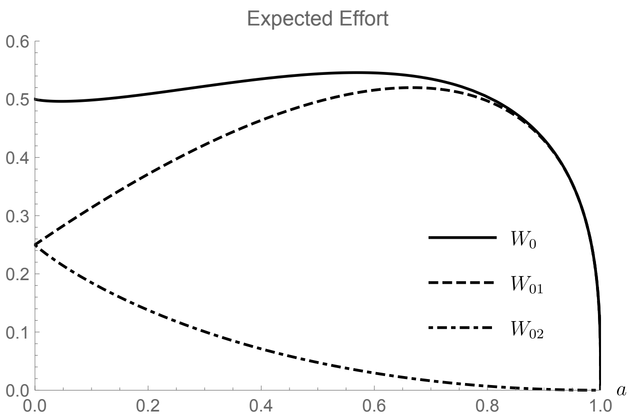

Figure 1: Total effort with different head starts

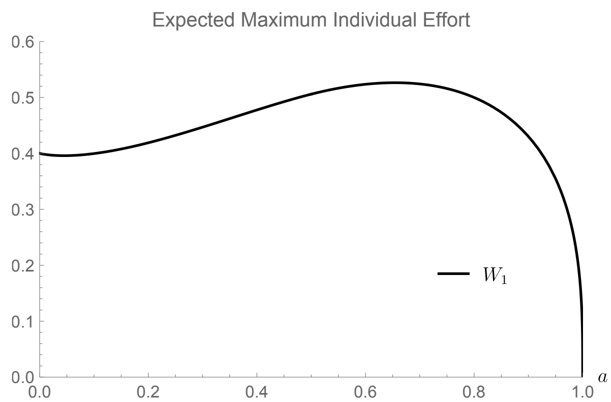

Figure 2: Maximum individual effort with different head starts

Example 4, in a similar vein, shows that "leveling the playing field" with handicaps may not be optimal when cost functions are not convex. It is, however, more challenging to construct such examples because a sufficient condition for Theorem 2 to hold is that $ec_i'(e)$ increases in e, which is weaker than the convexity of c_i.

Example 4. Consider a contest with two ex ante symmetric players with costs $c_i(e) = \frac{1}{2.3}\log(\log(50e + e_0))$, where e_0, the Euler's number, normalizes $c_i(0)$ to zero. Reaches for players are both approximately 429.2471. Consider headicapping Player 2 by a factor of $h \in (0, 1]$.

Let $W_0(h)$, $W_{01}(h)$ and $W_{02}(h)$ denote the expected total effort and individual expected effort. As Figure 3 shows, when h is very close to zero, which means Player 2 is extremely disadvantaged, Player 2 puts a lot of effort to stay competitive, leading to a higher expected total effort than that in the unbiased case ($h = 1$). For example, $h = 0.001$ induces the expected total effort of 54.0429, which is higher than 42.2403 induced by $h = 1$.

Figure 4 shows that the expected maximum individual effort, denoted by $W_1(h)$, exhibits a similar pattern: $h = 0.001$ induces the expected maximum individual effort of 54.0322, which is higher than 39.7861 induced by $h = 1$.

Therefore, "leveling the playing field" is not optimal in this setting. $\square$

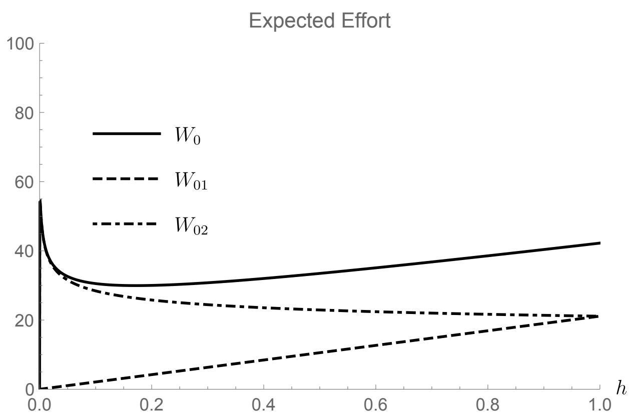

Figure 3: Total effort with different handicaps

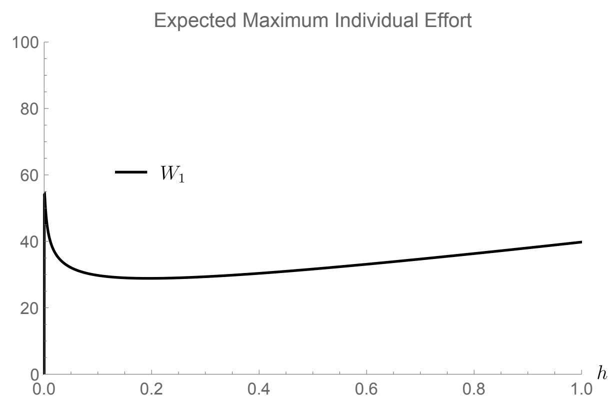

Figure 4: Maximizing individual effort with different handicaps

4 Optimal Favoritism with Both Instruments

One natural question to ask is: what happens if the organizer can use both instruments simultaneously? Similar questions have been studied in Kirkegaard [2012] and Franke et al. [2018]. They find that in revenue-maximizing all-pay auctions, it is generally optimal to combine the two instruments.

A general analysis of optimal combinations of the two instruments in all-pay contests with general objective functions is beyond the scope of this paper. In what follows, I concentrate on two most-studied objective functions, total effort and maximum individual effort, and demonstrate that the contest designer could benefit from using both instruments simultaneously.[6] For simplicity, and in line with Franke et al. [2018], I assume that the tie-breaking rule allocates the prize to Player 1 with probability 1.

Proposition 8. *When* $\Lambda(e_1, e_2) = \max\{e_1, e_2\}$ *and the contest designer is able to use both instruments simultaneously, it is optimal to set* $a = r_1$ *and* $h = 0$.

Proof. Given the tie-breaking rule and when $a = r_1$ and $h = 0$, the Nash equilibrium is in pure strategies: Player 1 chooses $e_1^* = r_1$ and wins the prize with certainty while Player 2 chooses $e_2^* = 0$. According to the definition of reaches, in equilibrium players choose effort strictly above reaches with probability zero. Therefore, inducing the expected maximum individual effort r_1 is the designer's first best result. □

In the optimum, the contest designer mimics a take-it-or-leave-it offer to the stronger player and achieves her first best result. For other tie-breaking rules, the contest designer could achieve a result arbitrarily close to the first best result. These results resonate with findings about optimal revenue-maximizing all-pay auctions in Franke et al. [2018].

Next, I turn my focus to total effort and revisit Example 1. It turns out that a take-it-or-leave-it offer is no longer optimal, but the simultaneous use of both instruments could still benefit the contest designer.

Example 1 (revisit) Recall that $\Lambda(e_1, e_2) = e_1 + e_2$, $c_1(e) = \frac{1}{2}e^2$ and $c_2(e) = e^2$. Reaches are $r_1 = \sqrt{2}$ and $r_2 = 1$. Notice that, in Proposition 8, the optimal combination of head starts and handicaps results in the same modified reaches for both players, and thus levels the playing field. In light of this, consider a combination (a, h) of instruments that levels the playing field, namely $r_1 = a + r_2 h$. Furthermore, assume $r_1 - r_2 \geq a \geq 0$.

When the playing field is leveled, both players get the expected payoff of zero in equilibrium.

[6] I thank an anonymous reviewer for suggesting Proposition 8.

Equilibrium strategies, denoted by $G_i^{HSHC}(e; a, h)$, are therefore:

$$G_1^{HSHC}(e; a, h) = \begin{cases} c_2(\frac{e-a}{h}), & \text{if } e \in [a, r_1), \\ 1, & \text{if } e \in [r_1, \infty), \end{cases}$$

$$G_2^{HSHC}(e; a, h) = \begin{cases} c_1(eh + a), & \text{if } e \in [0, r_2), \\ 1, & \text{if } e \in [r_2, \infty). \end{cases}$$

A take-it-or-leave-it offer, the optimal head start (alone) and the optimal handicap (alone) each induce the expected total effort 1.4145, 1.6213, and 1.6095, respectively.

Meanwhile, a combination $(a, h) = (\sqrt{2} - h^0, h^0)$, where $h^0 = \frac{1}{9}\left(3\sqrt{2} + 2\right) + \frac{2}{9}\sqrt{3\sqrt{2} + 1} \approx 1.2024$, induces the expected total effort approximately 1.6227.

Contrasted with Franke et al. [2018], a take-it-or-leave-it offer to the stronger player becomes suboptimal in this case. That is because cost functions are convex in effort, and, as a result, inducing the expected total effort equal to r_1 is no longer first best. Also, we can see that not every combination that levels the playing field is optimal. However, the contest designer could still benefit from the simultaneous use of both instruments. □

5 Conclusion

In this paper, I have analyzed the optimal favoritism in a complete-information, all-pay contest with two players, whose costs of effort are weakly convex. I find that any given player's effort distribution is ranked in the sense of first-order stochastic dominance according to how (ex post) symmetric the two players are. Consequently, it is optimal for the contest designer to "level the playing field" for any increasing objective functions, including total effort and maximum individual effort, as special cases. This result is a generalization of the conventional wisdom that a contest designer would prefer an unbiased contest when contestants are ex ante symmetric but a biased contest favoring the weaker player when the players are ex ante asymmetric. I then provide sufficient conditions for head starts to induce more effort than handicaps, which are consistent with findings in linear cost settings. Lastly, I consider two special cases wherein the contest designer's objective is either total effort or maximum individual effort. I find that the contest designer could benefit from the simultaneous use of both head starts and handicaps, and even achieve her first best result if her objective is maximum individual effort.

Appendices

A Proof of Proposition 2

When $a > r_1 - r_2$, by definition $T = r_1$, $w_1 = 0$ and $w_2 = 1 - c_2(T - a)$. Equilibrium strategies are:

$$G_1^{HS}(e;a) = \begin{cases} w_2, & \text{if } e \in [0,a), \\ c_2(e-a) + w_2, & \text{if } e \in [a,T), \\ 1, & \text{if } e \in [T,\infty), \end{cases}$$

$$G_2^{HS}(e;a) = \begin{cases} c_1(e+a), & \text{if } e \in [0,T-a), \\ 1, & \text{if } e \in [T-a,\infty). \end{cases}$$

Hence,

$$\frac{\partial G_1^{HS}(e;a)}{\partial a} = \begin{cases} c_2'(T-a) > 0, & \text{if } e \in [0,a), \\ c_2'(T-a) - c_2'(e-a) \geq 0, & \text{if } e \in [a,T), \\ 0, & \text{if } e \in [T,\infty), \end{cases}$$

$$\frac{\partial G_2^{HS}(e;a)}{\partial a} = \begin{cases} c_1'(e+a) > 0, & \text{if } e \in [0,T-a), \\ 0, & \text{if } e \in [T-a,\infty). \end{cases}$$

Similarly, when $0 < a < r_1 - r_2$, by definition $T = a + r_2$, $w_1 = 1 - c_1(T)$ and $w_2 = 0$. Equilibrium strategies are:

$$G_1^{HS}(e;a) = \begin{cases} 0, & \text{if } e \in [0,a), \\ c_2(e-a), & \text{if } e \in [a,T), \\ 1, & \text{if } e \in [T,\infty), \end{cases}$$

$$G_2^{HS}(e;a) = \begin{cases} c_1(e+a) + w_1, & \text{if } e \in [0,r_2), \\ 1, & \text{if } e \in [r_2,\infty). \end{cases}$$

Hence,

$$\frac{\partial G_1^{HS}(e;a)}{\partial a} = \begin{cases} 0, & \text{if } e \in [0,a), \\ -c_2'(e-a) < 0, & \text{if } e \in [a,T), \\ 0, & \text{if } e \in [T,\infty), \end{cases}$$

$$\frac{\partial G_2^{HS}(e;a)}{\partial a} = \begin{cases} c_1'(e+a) - c_1'(r_2+a) \leq 0, & \text{if } e \in [0,r_2), \\ 0, & \text{if } e \in [r_2,\infty). \end{cases}$$

All boundary cases can be shown as in the proof of Proposition 1.

B Proof of Proposition 3

When $1 < h < \frac{r_1}{r_2}$, by definition $T = r_2$, $w_1 = 1 - c_1(hT)$ and $w_2 = 0$. Equilibrium strategies are:

$$G_1^{HC}(e;h) = \begin{cases} c_2(\frac{e}{h}), & \text{if } e \in [0, hT), \\ 1, & \text{if } e \in [hT, \infty), \end{cases}$$

$$G_2^{HC}(e;h) = \begin{cases} w_1 + c_1(he), & \text{if } e \in [0, T), \\ 1, & \text{if } e \in [T, \infty). \end{cases}$$

Hence,

$$\frac{\partial G_1^{HC}(e;h)}{\partial h} = \begin{cases} -\frac{e}{h^2} c_2'(\frac{e}{h}) \leq 0, & \text{if } e \in [0, hT), \\ 0, & \text{if } e \in [hT, \infty), \end{cases}$$

$$\frac{\partial G_2^{HC}(e;h)}{\partial h} = \begin{cases} -Tc_1'(hT) + ec_1'(he) \leq 0, & \text{if } e \in [0, T), \\ 0, & \text{if } e \in [T, \infty). \end{cases}$$

Similarly, when $\frac{r_1}{r_2} < h$, by definition $T = \frac{r_1}{h}$, $w_1 = 0$ and $w_2 = 1 - c_2(T)$. Equilibrium strategies are:

$$G_1^{HC}(e;h) = \begin{cases} w_2 + c_2(\frac{e}{h}), & \text{if } e \in [0, r_1), \\ 1, & \text{if } e \in [r_1, \infty), \end{cases}$$

$$G_2^{HC}(e;h) = \begin{cases} c_1(he), & \text{if } e \in [0, T), \\ 1, & \text{if } e \in [T, \infty). \end{cases}$$

Hence,

$$\frac{\partial G_1^{HC}(e;h)}{\partial h} = \begin{cases} \frac{r_1}{h^2} c_2'(\frac{r_1}{h}) - \frac{e}{h^2} c_2'(\frac{e}{h}) \geq 0, & \text{if } e \in [0, r_1), \\ 0, & \text{if } e \in [r_1, \infty), \end{cases}$$

$$\frac{\partial G_2^{HC}(e;h)}{\partial h} = \begin{cases} e c_1'(he) \geq 0, & \text{if } e \in [0, T), \\ 0, & \text{if } e \in [T, \infty). \end{cases}$$

All boundary cases can be shown as in the proof of Proposition 1.

C Proof of Proposition 4

When $h < 1$, by definition $T = \tilde{r}_2 = r_2 h$, $w_1 = 1 - c_1(T)$ and $w_2 = 0$. Equilibrium strategies are:

$$G_1^{HC}(e;h) = \begin{cases} c_2(\frac{e}{h}), & \text{if } e \in [0,T), \\ 1, & \text{if } e \in [T,\infty), \end{cases}$$

$$G_2^{HC}(e;h) = \begin{cases} w_1 + c_1(eh), & \text{if } e \in [0,r_2), \\ 1, & \text{if } e \in [r_2,\infty). \end{cases}$$

Hence,

$$\frac{\partial G_1^{HC}(e;h)}{\partial h} = \begin{cases} -\frac{e}{h^2} c_2'(\frac{e}{h}) \leq 0, & \text{if } e \in [0,T), \\ 0, & \text{if } e \in [T,\infty), \end{cases}$$

$$\frac{\partial G_2^{HC}(e;h)}{\partial h} = \begin{cases} e c_1'(eh) - r_2 c_1'(T) \leq (e - r_2) c_1'(T) \leq 0, & \text{if } e \in [0,r_2), \\ 0, & \text{if } e \in [r_2,\infty). \end{cases}$$

All boundary cases can be shown as in the proof of Proposition 1.

D Proof of Proposition 5

Notice that:

$$G_1^{HS}(e;a^*) = \begin{cases} 0, & \text{if } e \in [0,a^*], \\ c_2(e-a^*), & \text{if } e \in [a^*, r_1], \\ 1, & \text{if } e > r_1, \end{cases} \quad G_1^{HC}(e;h^*) = \begin{cases} c_2(\frac{e}{h^*}), & \text{if } e \in [0, r_1] \\ 1, & \text{if } e > r_1. \end{cases}$$

$$G_2^{HS}(e;a^*) = \begin{cases} c_1(e+a^*), & \text{if } e \in [0,r_2], \\ 1, & \text{if } e > r_2, \end{cases} \quad G_2^{HC}(e;h^*) = \begin{cases} c_1(eh^*), & \text{if } e \in [0, r_2], \\ 1, & \text{if } e > r_2. \end{cases}$$

Because $e - a^* - \frac{e}{h^*} = e(1-\frac{1}{h^*}) - r_1(1-\frac{1}{h^*}) = (e-r_1)(1-\frac{1}{h^*}) \leq 0$, monotonicity of c_2 implies $c_2(e-a^*) \leq c_2(\frac{e}{h^*})$.

Similarly, $e + a^* - eh^* = e(1-h^*) - \frac{r_1}{h^*}(1-h^*) = (1-h^*)(e-r_2) \geq 0$, which implies $c_1(e+a^*) \geq c_1(eh^*)$.

Combining the two properties above, we can see that $G_1^{HS}(\cdot;a^*)$ FOSD $G_1^{HC}(\cdot;h^*)$ and that $G_2^{HC}(\cdot;h^*)$ FOSD $G_2^{HS}(\cdot;a^*)$ in the interior of each segment. FOSD holds also on boundaries because of the continuity of $G_i^{HS}(\cdot;a)$ and $G_i^{HC}(\cdot;h)$ in their first arguments.

E Proof of Proposition 6

Expected total effort induced by a^* is:

$$TE_a = \int_{a^*}^{r_1} e\mathrm{d}c_2(e-a^*) + \int_0^{r_2} e\mathrm{d}c_1(e+a^*) = r_1 + r_2 - \int_0^{r_2} c_2(e)\mathrm{d}e - \int_{a^*}^{r_1} c_1(e)\mathrm{d}e.$$

Expected total effort induced by h^* is:

$$TE_h = \int_0^{r_1} e\mathrm{d}c_2(\frac{e}{h^*}) + \int_0^{r_2} e\mathrm{d}c_1(eh^*) = r_1 + r_2 - h^* \int_0^{r_2} c_2(e)\mathrm{d}e - \frac{1}{h^*}\int_0^{r_1} c_1(e)\mathrm{d}e.$$

Therefore,

$$\begin{aligned} TE_a - TE_h &= h^* \int_0^{r_2} c_2(e)\mathrm{d}e + \frac{1}{h^*}\int_0^{r_1} c_1(e)\mathrm{d}e - \int_0^{r_2} c_2(e)\mathrm{d}e - \int_{a_2^*}^{r_1} c_1(e)\mathrm{d}e \\ &\geq (h^*-1)\int_0^{r_2} c_2(e)\mathrm{d}e - \frac{h^*-1}{h^*}\int_0^{r_1} c_1(e)\mathrm{d}e \\ &= \frac{h^*-1}{h^*}\int_0^{r_1} (c_2(\frac{e}{h^*}) - c_1(e))\mathrm{d}e \\ &\geq 0 \end{aligned}$$

F Proof of Proposition 7

Let $F^{HS}(\cdot;a)$ and $F^{HC}(\cdot;h)$ denote distributions of maximum individual effort induced by head starts and handicaps. We have

$$F^{HC}(e;h^*) = \begin{cases} c_2(\frac{e}{h^*})c_1(eh^*), & \text{if } e \in [0, r_2), \\ c_2(\frac{e}{h^*}), & \text{if } e \in [r_2, r_1], \\ 1, & \text{if } e > r_1. \end{cases}$$

$F^{HS}(\cdot;a^*)$ is more complicated and depends on the sizes of r_1 and r_2.

Case 1: $a^* = r_1 - r_2 \geq r_2$. In this case,

$$F^{HS}(e; a^*) = \begin{cases} 0, & \text{if } e \in [0, a^*], \\ c_2(e - a^*), & \text{if } e \in (a^*, r_1], \\ 1, & \text{if } e > r_1. \end{cases}$$

To show $F^{HS}(\cdot; a^*)$ FOSD $F^{HC}(\cdot; h^*)$, it suffices to prove $c_2(e - a^*) \leq c_2(\frac{e}{h^*})$ for $e \in [a^*, r_1]$. To see this, notice that $\frac{e}{h^*} - (e - a^*) = (e - r_1)(\frac{1}{h^*}) \geq 0$, monotonicity of c_2 implies $c_2(\frac{e}{h^*}) \geq c_2(e - a_2^*)$.

Case 2: $a^* < r_2$. In this case,

$$F^{HS}(e; a^*) = \begin{cases} 0, & \text{if } e \in [0, a^*], \\ c_2(e - a^*)c_1(e + a^*), & \text{if } e \in (a^*, r_2), \\ c_2(e - a^*), & \text{if } e \in [r_2, r_1], \\ 1, & \text{if } e > r_1. \end{cases}$$

To show $F^{HS}(\cdot; a^*)$ FOSD $F^{HC}(\cdot; h^*)$, in addition to the previous proof, we also need that $c_2(\frac{e}{h^*})c_1(eh^*) \geq c_2(e - a^*)c_1(e + a^*)$ for $e \in [a^*, r_2]$. To see this, notice that

$$\frac{c_2(\frac{e}{h^*})c_1(eh^*)}{c_2(e - a^*)c_1(e + a^*)} = \frac{\phi(e^2)}{\phi(e^2 - a^{*2})} \cdot \left(\frac{\frac{e}{h^*}}{e - a^*} \right)^m$$

On the right hand side, the first ratio is greater than one because $c_1(e_1)c_2(e_2) = \phi(e_1 e_2)e_2^m$ implies $\phi(\cdot)$ is increasing; the second ratio is also greater than one because $\frac{e}{h^*} \geq e - a^*$ and $m > 0$.

This completes the proof that $F^{HS}(\cdot; a^*)$ FOSD $F^{HC}(\cdot; h^*)$. As a property of FOSD, the expectation of the first distribution is larger.

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