# On optimal favoritism in all-pay contests

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**Abstract:** I analyze the optimal favoritism in a complete-information all-pay contest with two players, whose costs of effort are weakly convex. The contest designer could favor or harm some contestants using two instruments: head starts and handicaps. I find that equilibrium effort distributions are ranked according to how symmetric the two players are, in the sense of first-order stochastic dominance. Consequently, regardless of which instrument is used and what the designer's objective is, "levelling the playing field" is optimal.

**Keywords:** All-Pay Contests, Stochastic Dominance, Favoritism, Head Start, Handicap.

## 1 Introduction

Contests are widely used to allocate scarce resources among competing individuals. Examples include lobbying, college admissions, and competitions for job promotion opportunities (see Konrad [2009], Vojnović [2016], and Dechenaux et al. [2015]). In many situations, contestants are ex-ante asymmetric in their abilities and positions. For instance, when considering a job promotion competition, the manager may notice differences in productivity levels and progress among employees. Therefore, she may want to tailor the competition to encourage more effort from employees.

Problems like this are commonplace: the contest organizer often has discretionary power in designing contest rules and takes advantage of them to induce more competitions. Two general approaches are considered in the literature. One approach is to set individual-specific prizes as in Gürtler and Kräkel [2010] and Pérez-Castrillo and Wettstein [2016], where the contest reward depends on the identity of the winner. Another approach is to set individual-specific contest success functions. In this case, when facing the same bidding profile, the two players have different probabilities of winning; see Drugov and Ryvkin [2017] for a general analysis.

This note adopts the second approach and investigates the design of individual-specific contest success functions. Two commonly used instruments are considered: head starts, which are added

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to players' efforts, and handicaps, which discount players' efforts. In recent years, there has been growing literature exploring similar questions in various contest formats (see Konrad [2002], Epstein et al. [2011], Li and Yu [2012], Kirkegaard [2012], Franke et al. [2013], Seel and Wasser [2014], Kawamura and de Barreda [2014], and Franke et al. [2018]). The common wisdom suggests that it is optimal to "level the playing field": the contest designer prefers an unbiased contest when contestants are symmetric, but a biased contest favoring the weaker contestant when they are asymmetric. Notwithstanding, in most of the aforementioned literature (with the exception of Kawamura and de Barreda [2014] and Drugov and Ryvkin [2017]), costs are assumed to be linear in efforts. This assumption is not necessarily satisfied in many practical settings. For example, in a job promotion competition, it usually requires much more effort to improve the work quality from excellent to perfect than from mediocre to good for any given employee.

In light of this deficiency, this note examines the optimal design of biased contests when cost functions are weakly convex. The closest existing work is Drugov and Ryvkin [2017]. In that paper, the authors systematically study a similar problem by introducing a general class of biased contest success functions. When players are ex-ante symmetric, they provide conditions under which zero bias is optimal and prove by examples that biased contests could be optimal when such conditions fail. The key assumption in their model is that contest success functions are smooth; this includes Tullock [1980] lottery contests and Lazear and Rosen [1981] type tournaments. Nevertheless, one important class of contests is excluded by their assumption — that is, all-pay auctions, or more generally, all-pay contests.

In this note, I use the framework in Siegel [2014] and focus on the design of the optimal biased all-pay contest with complete information. A contest designer chooses the size of a head start or a handicap that is given to one of the two players, whose costs of effort are weakly convex. For simplicity, I omit the word "weakly" hereafter: all relations are in the weak sense unless explicitly stated as "strictly". Regardless of which instrument is used, I find that players' equilibrium strategies are ranked according to how symmetric they are, in the sense of first-order stochastic dominance. Consequently, for any objective functions increasing in efforts, it is optimal to "level the playing field". Comparing these two instruments, I show that no instrument can always dominate the other and provide sufficient conditions for head starts to be more efficient, under two most studied objective functions: aggregate effort and the highest effort.

The contribution of this note is threefold. First, by allowing for convex cost functions, my result generalizes the conventional wisdom that a contest designer benefits from "leveling the playing field" and that head starts are a more efficient tool than handicaps. Second, with two sets of examples, I demonstrate that it may not be optimal to "level the playing field" when cost functions are not convex and that sometimes head starts can be less efficient than handicaps. Finally, one implication of my result is that in a symmetric all-pay auction with complete information, the optimal head start is of size zero. This contrasts with the findings in Seel and Wasser [2014] that with incomplete information, the optimal head start is always of a strictly positive size. This discrepancy emphasizes the important role information plays in the design of all-pay contests and is consistent with findings in the design of the optimal handicaps in lottery contests (see Fu [2006] and Kirkegaard [2012] for analyses in complete and incomplete information settings respectively).

The remainder of the paper is organized as follows: Section 2 sets up the model; Section 3 identifies the optimal head start and handicap and makes a comparison between them; Section 4 discusses what happens when cost functions are not convex; and Section 5 concludes. Omitted proofs are in the appendix.

## 2 Model

There are two risk-neutral players and a risk-neutral contest designer. The players, indexed by $i = 1, 2$, compete for a single prize by exerting efforts $e_i \geq 0$. Each player is characterized by his valuation of the prize, $V_i > 0$, and a cost function (of effort) $c_i : \mathbb{R}_+ \to \mathbb{R}_+$.

The contest designer is characterized by a utility function $u(e_1, e_2)$, and she can influence the outcome of the contest with two instruments: head starts $a_i \geq 0$ and handicaps $h_i > 0$. Specifically, when player $i$ exerts effort $e_i$, her score is $s_i = a_i + h_i e_i$. Given $\mathbf{s} = (s_1, s_2)$, player $i$'s payoff is

$$u_i(\mathbf{s}) = P_i(\mathbf{s})V_i - \tilde{c}_i(s_i),$$

where $P_i : \mathbb{R}_+^2 \to [0, 1]$ is player $i$'s probability of winning, which satisfies

$$P_i(\mathbf{s}) = \begin{cases} 0, & \text{if } s_i < s_{-i}, \\ \text{any value in } [0,1], & \text{if } s_i = s_{-i}, \\ 1, & \text{if } s_i > s_{-i}, \end{cases}$$

such that $\sum_{j=1}^{2} P_j(\mathbf{s}) = 1$. $\tilde{c}_i(s_i)$ is player $i$'s cost of achieving $s_i$, in the presence of headstart $a_i$ and handicap $h_i$. That is,

$$\tilde{c}_i(s_i) = \begin{cases} 0, & \text{if } s_i \leq a_i, \\ c_i(\frac{s_i - a_i}{h_i}), & \text{otherwise.} \end{cases}$$

I make the following assumptions.

**Assumption 1.** $c_i(0) = 0$. *$c_i$ is strictly increasing, differentiable, and weakly convex.*

**Assumption 2.** $u(e_1, e_2)$ *is increasing in both arguments.*

The assumptions that $c_i(0) = 0$ and that $c_i$ is differentiable are for simplicity: all the results go through without them. The monotonicity assumptions on $c$ and $c'$ are, however, crucial. Examples 3 and 4 in Section 4 show that the main results may no longer hold when cost functions are not convex.

I normalize $V_i = 1$ for both players. This is without loss of generality. Extended from Siegel [2009], the equilibrium characterization uses the following definitions.

**Definitions:**

1. Player $i$'s *reach* $r_i$ is the maximum effort she can choose without obtaining a negative payoff if she wins the prize with certainty. That is, $r_i = c_i^{-1}(1)$.[1] Re-index players such that $r_1 \geq r_2$.

2. Player $i$'s *modified reach* $\tilde{r}_i$ is the maximum score she can obtain without incurring a negative payoff if she wins the prize with certainty, in the presence of handicaps and head starts. That is, $\tilde{r}_i = a_i + h_i r_i$.

3. The player with a lower modified reach is the *marginal player*.

4. The *threshold* $T$ of the contest is the modified reach of the marginal player: $T = \min\{a_i + h_i r_i\}$.

5. Player $i$'s *power* $w_i$ is her payoff when her score is $T$ and wins: $w_i = 1 - c_i(\frac{T - a_i}{h_i})$.[2]

A main departure from Siegel [2009] is the introduction of favoritism in the contests and, as a result, modified reaches. In my setting, $r_i$ determines player $i$'s ex-ante strength: without head starts or handicaps, the player with a higher reach obtains a positive equilibrium payoff and the player with a lower reach obtains an equilibrium payoff of zero. In the presence of head starts and handicaps, $\tilde{r}_i$ determines player $i$'s ex-post strength: the player with a higher modified reach obtains a positive equilibrium payoff and the player with a lower modified reach obtains an equilibrium payoff of zero. As a result, the marginal player and threshold in my setting are both determined by modified reaches.

With only two players, the model can be simplified because only the relative sizes of head starts and handicaps between them matter. It is without loss of generality to normalize the smaller $a_i$ to zero and the larger $h_i$ to one, resulting in the remaining $a_j$ being positive and $h_j$ between zero and one (these two $j$'s need not be the same). In the following analysis, I will focus on normalized head starts and handicaps: when I talk about $a_i > 0$, it implies $a_j = 0$; when I talk about $1 > h_i > 0$, it implies $h_j = 1$.

Lastly, notice that head starts and handicaps are substitutes in the design of biased contests. To avoid complications, I restrict the contest designer to use one of them, but not both simultaneously.

[1] I assume such $r_i$ exists. If this assumption is violated, player $i$'s cost of effort is always less than her value of prize. This leads to a trivial case, where she puts infinite effort in the contest.

[2] I assume that $T - a_i \geq 0$. If this assumption is violated, the contest is so biased that neither players has an incentive to compete in equilibrium, which is a trivial case.

# 3 Optimal Favoritism

## 3.1 Optimal Head Start

When handicaps are not available, $h_1 = h_2 = 1$. The relative size of head starts falls into one of the three cases:

Case 1: $a_1 > 0$. A head start is given to the ex-ante stronger player and makes her even stronger. By definition, $T = r_2$, $w_1 = 1 - c_1(T - a_1)$ and $w_2 = 0$

Case 2: $a_2 > 0$ and $r_2 + a_2 > r_1$. A large head start is given to the ex-ante weaker player such that ex-post she becomes the stronger player. By definition, $T = r_1$, $w_1 = 0$ and $w_2 = 1 - c_2(T - a_2)$.

Case 3: $a_2 > 0$ and $r_2 + a_2 \leq r_1$. A small head start is given to the ex-ante weaker player such that ex-post she is still the weaker player. By definition, $T = r_2 + a_2$, $w_1 = 1 - c_1(T)$ and $w_2 = 0$.

Below I illustrate the methodology in detail with Case 1, in which a head start is given to player 1. I first solve for the unique equilibrium and then present the first-order stochastic dominance (FOSD hereafter) property in Proposition 1. Proposition 2 is the parallel result when a head start is given to player 2, which corresponds to Cases 2 and 3.

Following Siegel [2009, 2014], the unique equilibrium is in mixed strategies. Let $G_i(e; a_1)$ denote the CDF of player $i$'s equilibrium strategy, which specifies player $i$'s probability of choosing effort less than or equal to $e$ when a head start of size $a_1$ is given to player 1. For simplicity, I call it player $i$'s *distribution induced by* $a_1$. Recall that $w_1 = 1 - c_1(T - a_1)$ and $w_2 = 0$. This payoff characterization pins down $G_i(e; a_1)$:

$$G_1(e; a_1) = \begin{cases} c_2(e + a_1), & \text{if } e \in [0, T - a_1), \\ 1, & \text{if } e \in [T - a_1, \infty), \end{cases}$$

$$G_2(e; a_1) = \begin{cases} w_1, & \text{if } e \in [0, a_1), \\ c_1(e - a_1) + w_1, & \text{if } e \in [a_1, T), \\ 1, & \text{if } e \in [T, \infty). \end{cases}$$

**Proposition 1.** *When a head start is given to player 1, for all $a_1' > a_1$, $G_i(\cdot; a_1)$ FOSD $G_i(\cdot; a_1')$, i.e. $G_i(e; a_1) \leq G_i(e; a_1')$ for $i = 1, 2$ and $e \geq 0$. In particular, for all $i = 1, 2$ and $a_1 \geq 0$, $G_i(\cdot; a_1^* = 0)$ FOSD $G_i(\cdot; a_1)$.*

*Proof.*

$$\frac{\partial G_1(e;a_1)}{\partial a_1} = \begin{cases} c_2'(e+a_1) > 0 & \text{if } e \in [0, T-a_1), \\ 0 & \text{if } e \in [T-a_1, \infty), \end{cases}$$

$$\frac{\partial G_2(e;a_1)}{\partial a_1} = \begin{cases} c_1'(T-a_1) > 0 & \text{if } e \in [0, a), \\ c_1'(T-a_1) - c_1'(e-a) \geq 0 & \text{if } e \in [a, T), \\ 0 & \text{if } e \in [T, \infty). \end{cases}$$

This shows that $G_i(\cdot;a_1)$ FOSD $G_i(\cdot;a_1')$ in the interior of each interval. FOSD holds on the boundaries as well due to $G_i(e;a_1)$'s continuity in $e$. □

In the proof, we see that a smaller head start induces more aggressive equilibrium strategies for both players in the sense of first-order stochastic dominance. There is, however, a subtle difference. Player 1 becomes more aggressive as long as $c_2$ is increasing. The main reason is that player 2 obtains a constant equilibrium payoff of zero. A smaller head start given to player 1 together with an increasing $c_2$ implies that player 2 can now compete with player 1 at a lower cost. To keep player 2's payoff constant, player 1 becomes more aggressive.

The analogous property of player 2's strategy, however, requires more assumptions on $c_1$. This is because player 1's equilibrium payoff $w_1$ decreases as a result of a smaller head start. On one hand, the decrease implies that player 2 now competes more aggressively. On the other hand, similar to the previous analysis, a smaller head start given to player 1 together with an increasing $c_1$ implies that player 1 will now compete with player 2 at a higher cost. Thus, player 2 becomes less aggressive. The sign of the overall effect, captured by $c_1'(T-a_1) - c_1'(e-a)$, is determined by the curvature of $c_1$. To be more specific, when $c_1$ is increasing and convex, a smaller head start given to player 1 makes player 2 compete more aggressively.

Combining the analyses above, Proposition 1 concludes that the contest designer would be worse off if she gives a strictly positive head start to the ex-ante stronger player. As Proposition 2 suggests, this idea of "leveling the playing field" also applies when the designer gives a head start to the ex-ante weaker player.

**Proposition 2.** *When a head start is given to player 2, that is, $a_2 > 0$:*

1. *For all $a_2' > a_2 \geq r_1 - r_2$ and $i = 1, 2$, $G_i(\cdot;a_2)$ FOSD $G_i(\cdot;a_2')$.*
2. *For all $r_1 - r_2 \geq a_2' > a_2$ and $i = 1, 2$, $G_i(\cdot;a_2')$ FOSD $G_i(\cdot;a_2)$.*
3. *Therefore, for all $a_2 \geq 0$ and $i = 1, 2$, $G_i(\cdot;a_2^* = r_1 - r_2)$ FOSD $G_i(\cdot;a_2)$.*

Lemma 1 combines Propositions 1 and 2 and shows that the equilibrium effort distributions induced by $a_2^*$ dominate those induced by any other head starts.

**Lemma 1.** *$G_i(\cdot; a_2^*)$ FOSD $G_i(\cdot; a_k)$ for all $i, k \in \{1, 2\}$.*

*Proof.* According to Proposition 1 $G_i(\cdot; a_1)$ is dominated by $G_i(\cdot; a_1^* = 0)$, which is the same as $G_i(\cdot; a_2 = 0)$. By Proposition 2, $G_i(\cdot; a_2 = 0)$ is dominated by $G_i(\cdot; a_2^*)$. Thus, $G_i(\cdot; a_2^*)$ FOSD $G_i(\cdot; a_1)$. Moreover, Proposition 2 shows that $G_i(\cdot; a_2^*)$ FOSD any $G_i(\cdot; a_2)$. □

Because $G_1(\cdot)$ and $G_2(\cdot)$ are independent, FOSD properties in Lemma 1 imply that it is optimal to set $a_2^* = r_1 - r_2$. More formally, let $E_i^{ak}$ be a random variable from $G_i(\cdot; a_k)$ and recall that the designer's objective function is $u(e_1, e_2)$, where $e_i$ is a realization of $E_i$. The following theorem is the main result in this section.

**Theorem 1.** *$\mathbb{E}u(E_1^{ak}, E_2^{ak})$ is maximized when $a^* = r_1 - r_2$ and $k = 2$.*

*Proof.* That $E_i^{a^*2}$ FOSD $E_i^{ak}$ for $i, k = 1, 2$ together with the independence between $E_1^{ak}$ and $E_2^{ak}$ implies that $(E_1^{a^*2}, E_2^{a^*2})$ FOSD $(E_1^{ak}, E_2^{ak})$. Therefore, for any increasing objective function $u(e_1, e_2)$,

$$\mathbb{E}u(E_1^{ak}, E_2^{ak}) \leq \mathbb{E}u(E_1^{a^*2}, E_2^{a^*2})$$

□

Theorem 1 shows that giving a head start of size $a^* = r_1 - r_2$ to player 2, namely "leveling the playing field", maximizes the expectation of any increasing utility functions, in which $u(e_1, e_2) = e_1 + e_2$ and $u(e_1, e_2) = \max\{e_1, e_2\}$ are included as special cases. Applying Theorem 1, Corollaries 1 and 2 consider these utility functions and characterize optimal head starts in an ex-ante asymmetric contest and in an ex-ante symmetric contest respectively.

**Corrolary 1.** *In an ex-ante asymmetric contest, providing a head start of size $r_1 - r_2$ to player 2 maximizes both expected aggregate effort and the expected highest effort.*

**Corrolary 2.** *In an ex-ante symmetric contest, zero head start maximizes both the expected aggregate effort and the expected highest effort.*

## 3.2 Optimal Handicap

When head starts are not available, $a_1 = a_2 = 0$. The analysis of the optimal handicap is similar to that of the optimal head start. Hence I list below the corresponding propositions and the main result and leave details in the appendix.

**Proposition 3.** *When player 1 is handicapped, that is, $1 \geq h_1 > 0$:*

1. *For all $h_1' > h_1 \geq \frac{r_2}{r_1}$ and $i = 1, 2$, $G_i(\cdot; h_1)$ FOSD $G_i(\cdot; h_1')$.*
2. *For all $\frac{r_2}{r_1} \geq h_1' > h_1$ and $i = 1, 2$, $G_i(\cdot; h_1')$ FOSD $G_i(\cdot; h_1)$.*

*3. Therefore, for all $1 \geq h_1 > 0$ and $i = 1, 2$, $G_i(\cdot; h_1^* = \frac{r_2}{r_1})$ FOSD $G_i(\cdot; h_1)$.*

**Proposition 4.** *When player 2 is handicapped, for all $1 \geq h_2 > h_2' > 0$ and $i = 1, 2$, $G_i(\cdot; h_2)$ FOSD $G_i(\cdot; h_2')$. In particular, for all $i = 1, 2$ and $1 \geq h_2 > 0$, $G_i(\cdot; h_2^* = 1)$ FOSD $G_i(\cdot; h_2)$.*

**Theorem 2.** *$\mathbb{E}u(E_1^{hk}, E_2^{hk})$ is maximized when $h^* = \frac{r_2}{r_1}$ and $k = 1$.*

**Corrolary 3.** *In an ex-ante asymmetric contest, handicapping player 1 by the size of $\frac{r_2}{r_1}$ maximizes both expected aggregate effort and the expected highest effort.*

**Corrolary 4.** *In an ex-ante symmetric contest, zero handicapping maximizes both the expected aggregate effort and the expected highest effort.*

**Remarks:** all results in this section still holds when $ec_i'(e)$ is increasing in $e$, which is implied by convexity of $c_i$. More discussion about this can be found in Section 4.

### 3.3 Comparison between Instruments

If the contest organizer could choose between the two instruments, which one would she choose? Proposition 5 shows that in general there is no definite answer: player 1 exerts more effort when head starts are used instead of handicaps; however, player 2 exerts less effort in this case.

**Proposition 5.** *$G_1(\cdot; a_2^*)$ FOSD $G_1(\cdot; h_1^*)$, but $G_2(\cdot; h_1^*)$ FOSD $G_2(\cdot; a_2^*)$.*

Proposition 5 implies that a contest designer who values both players' efforts does not always prefer one instrument over the other. This is illustrated with Examples 1 and 2 below of two widely used objective functions: $u(e_1, e_2) = e_1 + e_2$ and $u(e_1, e_2) = \max\{e_1, e_2\}$.

**Example 1.** Suppose that $u(e_1, e_2) = e_1 + e_2$.

When $c_1(e) = \frac{1}{2}e$ and $c_2(e) = e^2$, reaches are $r_1 = 2$ and $r_2 = 1$. Optimal instruments are $a_2^* = 1$ and $h_1^* = \frac{1}{2}$. The expected aggregate effort induced by the optimal head start and the optimal handicap are $\frac{23}{12}$ and $\frac{22}{12}$, respectively. The contest organizer therefore prefers "leveling the playing field" with head starts.

When $c_1(e) = \frac{1}{2}e^2$ and $c_2(e) = e^3$, reaches are $r_1 = \sqrt{2}$ and $r_2 = 1$. Optimal instruments are $a_2^* = \sqrt{2} - 1$ and $h_1^* = \frac{1}{\sqrt{2}}$. The expected aggregate effort induced by the optimal head start and the optimal handicap are $\sqrt{2} + \frac{1}{\sqrt{2}} - \frac{5}{12} \approx 1.7047$ and $\frac{2}{3} + \frac{3}{2\sqrt{2}} \approx 1.7273$, respectively. The contest organizer therefore prefers "leveling the playing field" with handicaps. □

**Example 2.** Suppose that $u(e_1, e_2) = \max\{e_1, e_2\}$.

When $c_1(e) = e$ and $c_2(e) = \frac{9}{8}e$, reaches are $r_1 = 1$ and $r_2 = \frac{8}{9}$. Optimal instruments are $a_2^* = \frac{1}{9}$ and $h_1^* = \frac{8}{9}$. The expected highest efforts induced by the optimal head start and the optimal handicap are $\frac{2503}{3888} \approx 0.6438$ and $\frac{307}{486} \approx 0.6317$, respectively. The contest organizer therefore prefers "leveling the playing field" with head starts.

When $c_1(e) = e^3$ and $c_2(e) = \frac{9}{8}e$, reaches are $r_1 = 1$ and $r_2 = \frac{8}{9}$. Optimal instruments are $a_2^* = \frac{1}{9}$ and $h_1^* = \frac{8}{9}$. The expected highest efforts induced by the optimal head start and the optimal handicap are $\frac{24056}{32805} \approx 0.7333$ and $\frac{199}{270} \approx 0.7370$, respectively. The contest organizer therefore prefers "leveling the playing field" with handicaps. □

Despite the lack of a general ranking result, Propositions 6 and 7 provide sufficient conditions under which head starts are preferred.

**Proposition 6.** *If $c_1(e) \leq c_2(h_1^* e)$ for all $e \in [0, r_1]$, then the contest with the optimal head start induces higher expected aggregate effort than that with the optimal handicap.*

**Proposition 7.** *If there is a function $\phi(\cdot) : [0, r_1 r_2] \to \mathbb{R}_+$ and $m \geq 0$ such that for all $e_1 \in [0, r_1]$ and $e_2 \in [0, r_2]$, $c_1(e_1)c_2(e_2) = \phi(e_1 e_2)e_2^m$, then the contest with the optimal head start induces higher the expected highest effort than that with the the optimal handicap.*

Proposition 6 suggests that when $c_1$ is lower than $c_2$ even if the handicap is counted in, the optimal head start induce higher expected aggregate effort than the handicap does. With a much lower cost, player 1's effort is of higher magnitude compared with player 2's. Recall that when head starts are used instead of handicaps, player 1's expected effort gets higher, but player 2's expected effort gets lower. The increase in player 1's expected effort, therefore, dominates the decrease in player 2's, leading to an increase in aggregate effort. The condition in Proposition 6 is quite general. For example, it is satisfied when $c_1$ is a scale down of $c_2$, namely $c_1(e) = \tau_0 c_2(e)$ for some $\tau_0 < 1$. This includes linear cost functions like the ones in Li and Yu [2012], as special cases.

Likewise, for the highest effort, Proposition 7 suggests that when $c_2$ is of higher degrees than $c_1$, the optimal head start is more efficient. The main reason is that the distribution of an order statistic is the multiplication of two players' equilibrium strategies. A sufficient condition for head starts to be more efficient is $c_1(e + a_2^*)c_1(e - a_2^*) \leq c_1(\frac{e}{h_1^*})c_2(eh_1^*)$, which holds when the condition in Proposition 7 is met. This condition is also quite general. For example, it is satisfied for monomial cost functions satisfying $\text{Deg}(c_1) \leq \text{Deg}(c_2)$. This includes linear cost functions, which are widely studied in the literature, as special cases.

# 4 Non-Convex Cost Functions

What would happen if cost functions are not convex? Example 3 shows that, in this case, "leveling the playing field" with head starts may not be optimal.

**Example 3.** In a contest with two ex-ante symmetric players with costs $c_i(e) = \sqrt[3]{e}$, consider giving player 1 a head start of size $a_1$. In equilibrium, expected efforts of players 1 and 2 are $W_{01}(a_1) = \frac{1}{4}\left(3a_1^{4/3} - 4a_1 + 1\right)$ and $W_{02}(a_1) = \frac{\sqrt[3]{1-a_1}}{4}(3a_1 + 1)$ respectively. Figure 1 depicts both individual efforts and aggregate effort. Expected aggregate effort $W_0(a_1)$ attains its maximum at $a_1^* \approx 0.56882$.

Let $W_1(a_1)$ denote the expected highest effort. $W_1(a_1) = \frac{1}{4}(-3(\sqrt[3]{1-a_1}\ -1)a_1^{4/3} + (7\sqrt[3]{1-a_1} - 4)a_1 + 1)$ is maximized at $a_1^* \approx 0.65455$, as in Figure 2.

Both maximizers for expected aggregate effort and the expected highest effort are strictly positive when two players are ex-ante symmetric. Hence "levelling the playing field" is not optimal in this setting. □

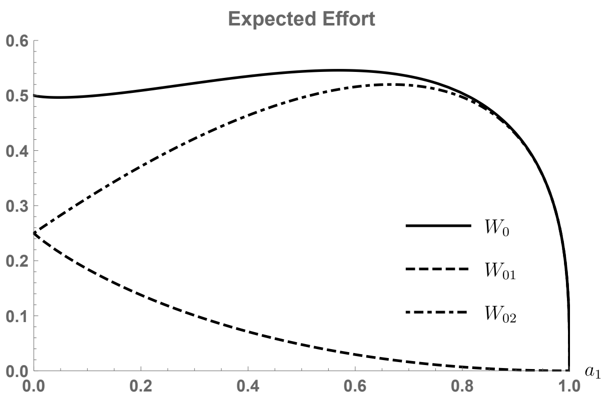


Figure 1: Expected aggregate effort

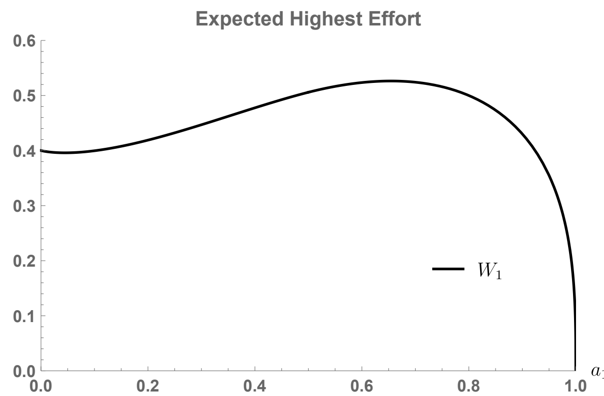


Figure 2: Expected Highest Effort

Similarly, Example 4 shows that "leveling the playing field" with handicaps may not be optimal when cost functions are not convex. It is, however, more challenging to construct such examples because a sufficient condition for Theorem 2 to hold is that $ec_i'(e)$ increases in $e$, which is weaker than the convexity of $c_i$.

**Example 4.** Consider a contest with two ex-ante symmetric players with costs $c_i(e) = \frac{1}{2.3}\log(\log(50e+e_0))$, where $e_0$, the Euler's number, normalizes $c_i(0)$ to zero. Reaches for players are both approximately 429.247. Consider headicapping player 2 by a factor of $h_2$.

Let $W_0(h_2)$, $W_{01}(h_2)$ and $W_{02}(h_2)$ denote the expected aggregate effort and individual expected effort. As Figure 3 shows, when $h_2$ is very close to zero, which means player 2 is extremely disadvantaged, player 2 puts a lot of effort to stay competitive, leading to a higher expected aggregate effort than that in the unbiased case ($h_2 = 1$). To be more precise, $h_2^* \approx 0.00076$ induces the expected aggregate effort of 54.3252, which is higher than 42.2403 induced by $h_2 = 1$.

Figure 4 shows that the expected highest effort, denoted by $W_1(h_2)$, exhibits a similar pattern: $h_2^* \approx 0.00076$ induces the expected highest effort of 54.3177, which is higher than 39.7861 induced by $h_2 = 1$.

Both maximizers for expected aggregate effort and the expected highest effort are strictly less than one when two players are ex-ante symmetric. Hence "levelling the playing field" is not optimal in this setting. □

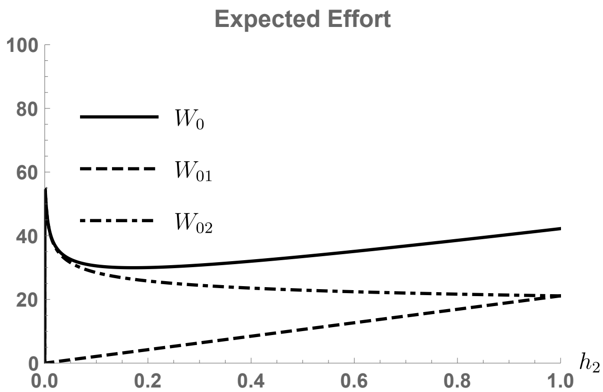


Figure 3: Expected aggregate effort

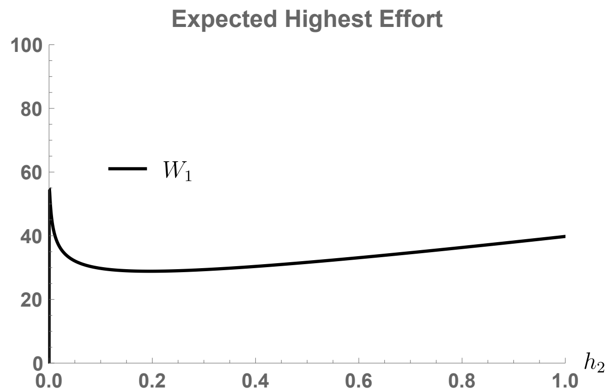


Figure 4: Expected Highest Effort

# 5 Conclusion

In this note, I have analyzed the optimal favoritism in contests with complete information. The first contribution is to introduce a framework that studies the optimal favoritism in all-pay contests, which includes all-pay auctions as a special case. The analysis is parallel to Kirkegaard [2012]'s analysis in the incomplete information setting and complements Drugov and Ryvkin [2017]'s general analysis in contests with continuous contest success functions. I find that contestants' equilibrium effort distributions are ranked according to how symmetric they are, in the sense of first-order stochastic dominance. Consequently, it is optimal for the contest designer to "level the playing field" for any increasing objective functions, including aggregate effort and the highest effort as special cases. This result is a generalization of the conventional wisdom that a contest designer would prefer an unbiased contest when contestants are ex-ante symmetric, but a biased contest favoring the weaker player when they are ex-ante asymmetric. Moreover, I find sufficient conditions for head starts to induce more effort than handicaps, which are consistent with findings in linear cost settings. The second contribution is two sets of examples illustrating how the conventional wisdom may fail when convexity of costs or the proposed conditions are not satisfied. Lastly, an implication of my result is that in a symmetric all-pay auction with complete information, it is optimal to set zero head start. This contrasts with the findings in Seel and Wasser [2014] in an incomplete information setting. The discrepancy emphasizes the important role of information in the design of the optimal contest.

# Appendices

## A Proof of Proposition 2

When $a_2 > r_1 - r_2$, by definition $T = r_1$, $w_1 = 0$ and $w_2 = 1 - c_2(T - a_2)$. Equilibrium strategies are:

$$G_1(e; a_2) = \begin{cases} w_2, & \text{if } e \in [0, a_2), \\ c_2(e - a_2) + w_2, & \text{if } e \in [a_2, T), \\ 1, & \text{if } e \in [T, \infty), \end{cases}$$

$$G_2(e; a_2) = \begin{cases} c_1(e + a_2), & \text{if } e, \in [0, T - a_2), \\ 1, & \text{if } e \in [T - a_2, \infty). \end{cases}$$

Hence,

$$\frac{\partial G_1(e; a_2)}{\partial a_2} = \begin{cases} c_2'(T - a_2) > 0, & \text{if } e \in [0, a_2), \\ c_2'(T - a_2) - c_2'(e - a_2) \geq 0, & \text{if } e \in [a_2, T), \\ 0, & \text{if } e \in [T, \infty), \end{cases}$$

$$\frac{\partial G_2(e; a_2)}{\partial a_2} = \begin{cases} c_1'(e + a_2) > 0, & \text{if } e \in [0, T - a_2), \\ 0, & \text{if } e \in [T - a_2, \infty). \end{cases}$$

Part 1 of the proposition is proved.

Similarly, when $a_2 < r_1 - r_2$, by definition $T = a_2 + r_2$, $w_1 = 1 - c_1(T)$ and $w_2 = 0$. Equilibrium strategies are:

$$G_1(e; a_2) = \begin{cases} 0, & \text{if } e \in [0, a_2), \\ c_2(e - a_2), & \text{if } e \in [a_2, T), \\ 1, & \text{if } e \in [T, \infty), \end{cases}$$

$$G_2(e; a_2) = \begin{cases} c_1(e + a_2) + w_1, & \text{if } e \in [0, r_2), \\ 1, & \text{if } e \in [r_2, \infty). \end{cases}$$

Hence,

$$\frac{\partial G_1(e;a_2)}{\partial a_2} = \begin{cases} 0, & \text{if } e \in [0, a_2), \\ -c_2'(e-a_2) < 0, & \text{if } e \in [a_2, T), \\ 0, & \text{if } e \in [T, \infty), \end{cases}$$

$$\frac{\partial G_2(e;a_2)}{\partial a_2} = \begin{cases} c_1'(e+a_2) - c_1'(r_2+a_2) \leq 0, & \text{if } e \in [0, T-a_2), \\ 0, & \text{if } e \in [T-a_2, \infty). \end{cases}$$

This completes the proof of part 2. Part 3 is immediate from the first two results.

## B Proof of Proposition 3

When player 1 is handicapped and $h_1 > \frac{r_2}{r_1}$, by definition $T = r_2$, $w_1 = 1 - c_1(\frac{T}{h_1})$ and $w_2 = 0$. Equilibrium strategies are:

$$G_1(e;h_1) = \begin{cases} c_2(eh_1), & \text{if } e \in [0, \frac{T}{h_1}), \\ 1, & \text{if } e \in [\frac{T}{h_1}, \infty), \end{cases}$$

$$G_2(e;h_1) = \begin{cases} w_1 + c_1(\frac{e}{h_1}), & \text{if } e \in [0, T), \\ 1, & \text{if } e \in [T, \infty). \end{cases}$$

Hence,

$$\frac{\partial G_1(e;h_1)}{\partial h_1} = \begin{cases} ec_2'(eh_1) \geq 0, & \text{if } e \in [0, \frac{T}{h_1}), \\ 0, & \text{if } e \in [\frac{T}{h_1}, \infty), \end{cases}$$

$$\frac{\partial G_2(e;h_1)}{\partial h_1} = \begin{cases} \frac{T}{h_1^2}c_1'(\frac{T}{h_1}) - \frac{e}{h_1^2}c_1'(\frac{e}{h_1}) \geq 0, & \text{if } e \in [0, T), \\ 0, & \text{if } e \in [T, \infty). \end{cases}$$

This shows that when $h_1' > h_1 \geq \frac{r_2}{r_1}$, $G_i(\cdot;h_1)$ FOSD $G_i(\cdot;h_1')$.

When player 1 is handicapped and $h_1 < \frac{r_2}{r_1}$, by definition $T = \tilde{r}_1 = r_1 h_1$, $w_1 = 0$ and $w_2 = 1 - c_2(T)$. Equilibrium strategies are:

$$G_1(e;h_1) = \begin{cases} w_2 + c_2(eh_1), & \text{if } e \in [0, r_1), \\ 1, & \text{if } e \in [r_1, \infty), \end{cases}$$

$$G_2(e;h_1) = \begin{cases} c_1(\frac{e}{h_1}), & \text{if } e \in [0, T), \\ 1, & \text{if } e \in [T, \infty). \end{cases}$$

Hence,

$$\frac{\partial G_1(e;h_1)}{\partial h_1} = \begin{cases} ec_2'(eh_1) - r_1 c_2'(T) \leq (e - r_1)c_2'(T) \leq 0, & \text{if } e \in [0, r_1), \\ 0, & \text{if } e \in [r_1, \infty), \end{cases}$$

$$\frac{\partial G_2(e;h_1)}{\partial h_1} = \begin{cases} -\frac{e}{h_1^2} c_1'(\frac{e}{h_1}) \leq 0, & \text{if } e \in [0, T), \\ 0, & \text{if } e \in [T, \infty). \end{cases}$$

This shows that when $\frac{r_2}{r_1} \geq h_1' > h_1$, $G_i(\cdot; h_1')$ FOSD $G_i(\cdot; h_1)$.

## C Proof of Proposition 4

When player 2 is handicapped, by definition $T = \tilde{r}_2 = r_2 h_2$, $w_1 = 1 - c_1(T)$ and $w_2 = 0$. Equilibrium strategies are:

$$G_1(e;h_2) = \begin{cases} c_2(\frac{e}{h_2}), & \text{if } e \in [0, T), \\ 1, & \text{if } e \in [T, \infty), \end{cases}$$

$$G_2(e;h_2) = \begin{cases} w_1 + c_1(eh_2), & \text{if } e, \in [0, r_2), \\ 1, & \text{if } e \in [r_2, \infty). \end{cases}$$

Hence,

$$\frac{\partial G_1(e;h_2)}{\partial h_2} = \begin{cases} -\frac{e}{h_2^2} c_2'(\frac{e}{h_2}) \leq 0, & \text{if } e \in [0, T), \\ 0, & \text{if } e \in [T, \infty), \end{cases}$$

$$\frac{\partial G_2(e;h_2)}{\partial h_2} = \begin{cases} ec_1'(eh_2) - r_2 c_1'(T) \leq (e - r_2)c_1'(T) \leq 0, & \text{if } e \in [0, r_2), \\ 0, & \text{if } e \in [r_2, \infty). \end{cases}$$

Therefore for all $i = 1, 2$ and $1 \geq h_2 > h_2' > 0$, $G_i(\cdot; h_2)$ FOSD $G_i(\cdot; h_2')$.

## D Proof of Proposition 5

Notice that:

$$G_1(e;a_2^*) = \begin{cases} 0, & \text{if } e \in [0, a_2^*], \\ c_2(e - a_2^*), & \text{if } e \in [a_2^*, r_1], \\ 1, & \text{if } e > r_1, \end{cases} \quad G_1(e;h_1^*) = \begin{cases} c_2(eh_1^*), & \text{if } e \in [0, r_1] \\ 1, & \text{if } e > r_1. \end{cases}$$

$$G_2(e;a_2^*) = \begin{cases} c_1(e + a_2^*), & \text{if } e \in [0, r_2], \\ 1, & \text{if } e > r_2, \end{cases} \quad G_2(e;h_1^*) = \begin{cases} c_1(\frac{e}{h_1^*}), & \text{if } e \in [0, r_2], \\ 1, & \text{if } e > r_2. \end{cases}$$

Because $e - a_2^* - eh_1^* = e(1 - h_1^*) - r_1(1 - h_1^*) = (e - r_1)(1 - h_1^*) \leq 0$, monotonicity of $c_2$ implies $c_2(e - a_2^*) \leq c_2(eh_1^*)$.

Similarly, $e + a_2^* - \frac{e}{h_1^*} = \frac{e}{h_1^*}(h_1^* - 1) + r_1(1 - h_1^*) = (1 - h_1^*)(r_1 - \frac{e}{h_1^*}) \geq 0$, which implies $c_1(e + a_2^*) \geq c_1(\frac{e}{h_1^*})$.

Combining the two properties above, we can see that $G_1(\cdot; a_2^*)$ FOSD $G_1(\cdot; h_1^*)$ and that $G_2(\cdot; h_1^*)$ FOSD $G_2(\cdot; a_2^*)$ in the interior of each segment. Just like the proof of Proposition 1, FOSD holds also on boundaries because of the continuity of $G_i(\cdot; a_2)$ and $G_i(\cdot; h_1)$ in their first arguments.

## E Proof of Proposition 6

Expected aggregate effort induced by $a_2^*$ is:

$$TE_a = \int_{a_2^*}^{r_1} e\mathrm{d}c_2(e - a_2^*) + \int_0^{r_2} e\mathrm{d}c_1(e + a_2^*) = r_1 + r_2 - \int_0^{r_2} c_2(e)\mathrm{d}e - \int_{a_2^*}^{r_1} c_1(e)\mathrm{d}e.$$

Expected aggregate effort induced by $h_1^*$ is:

$$TE_h = \int_0^{r_1} e\mathrm{d}c_2(eh_1^*) + \int_0^{r_2} e\mathrm{d}c_1(\frac{e}{h_1^*}) = r_1 + r_2 - \frac{1}{h_1^*}\int_0^{r_2} c_2(e)\mathrm{d}e - h_1^*\int_0^{r_1} c_1(e)\mathrm{d}e.$$

Therefore,

$$\begin{aligned} \Delta = TE_a - TE_h &= \frac{1}{h_1^*}\int_0^{r_2} c_2(e)\mathrm{d}e + h_1^*\int_0^{r_1} c_1(e)\mathrm{d}e - \int_0^{r_2} c_2(e)\mathrm{d}e - \int_{a_2^*}^{r_1} c_1(e)\mathrm{d}e \\ &\geq \frac{1 - h_1^*}{h_1^*}\int_0^{r_2} c_2(e)\mathrm{d}e - (1 - h_1^*)\int_0^{r_1} c_1(e)\mathrm{d}e \\ &= (1 - h_1^*)\int_0^{r_1} (c_2(h_1^* e) - c_1(e))\mathrm{d}e \geq 0 \end{aligned}$$

## F Proof of Proposition 7

Let $F(\cdot; a_2^*)$ and $F(\cdot; h_1^*)$ denote distributions of the highest effort induced by optimal instruments. We have

$$F(e; h_1^*) = \begin{cases} c_2(eh_1^*)c_1(\frac{e}{h_1^*}), & \text{if } e \in [0, r_2), \\ c_2(eh_1^*), & \text{if } e \in [r_2, r_1], \\ 1, & \text{if } e > r_1. \end{cases}$$

$F(\cdot; a_2^*)$ is more complicated and depends on the sizes of $r_1$ and $r_2$. When $a_2^* = r_1 - r_2 \geq r_2$,

$$F(e; a_2^*) = \begin{cases} 0, & \text{if } e \in [0, a_2^*), \\ c_2(e - a_2^*), & \text{if } e \in [a_2^*, r_1], \\ 1, & \text{if } e > r_1. \end{cases}$$

When $a_2^* < r_2$,

$$F(e; a_2^*) = \begin{cases} 0, & \text{if } e \in [0, a_2^*), \\ c_2(e - a_2^*)c_1(e + a_2^*), & \text{if } e \in [a_2^*, r_2), \\ c_2(e - a_2^*), & \text{if } e \in [r_2, r_1], \\ 1, & \text{if } e > r_1. \end{cases}$$

Because $eh_1^* - (e - a_2^*) = e(h_1^* - 1) + r_1(1 - h_1^*) = (r_1 - e)(1 - h_1^*) \geq 0$, monotonicity of $c_2$ implies $c_2(eh_1^*) \geq c_2(e - a_2^*)$.

To see $c_2(eh_1^*)c_1(\frac{e}{h_1^*}) \geq c_2(e - a_2^*)c_1(e + a_2^*)$, notice that

$$\frac{c_2(eh_1^*)c_1(\frac{e}{h_1^*})}{c_2(e - a_2^*)c_1(e + a_2^*)} = \frac{\phi(e^2)}{\phi(e^2 - a_2^{*2})} \cdot \left(\frac{eh_1^*}{e - a_2^*}\right)^m$$

On the right hand side, the first ratio is greater than one because $c_1(e_1)c_2(e_2) = \phi(e_1e_2)e_2^m$ implies $\phi(\cdot)$ is increasing; the second ratio is also greater than one because $eh_1^* \geq e - a_2^*$ and $m > 0$.

This completes the proof that $F(\cdot; a_2^*)$ FOSD $F(\cdot; h_1^*)$. As a property of FOSD, the expectation of the first distribution is larger.

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